Math 2550
Homework 8
Solutions

HW 8 Solutions

① (a) $A = \begin{pmatrix} 3 & 0 \\ 8 & -1 \end{pmatrix}$

<u>characteristic polynomial</u>

$$\det(A - \lambda I_2) = \det\left(\begin{pmatrix} 3 & 0 \\ 8 & -1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)$$

$$= \det \begin{pmatrix} 3-\lambda & 0 \\ 8 & -1-\lambda \end{pmatrix}$$

$$= (3-\lambda)(-1-\lambda) - (0)(8)$$

$$= (3-\lambda)(-1-\lambda)$$

$$= (\lambda-3)(\lambda+1)$$

I factored out two (-1)'s and they cancelled each other out

Thus, $\lambda = 3, -1$ are the eigenvalues of A.

$\lambda = 3$ has algebraic multiplicity 1.

$\lambda = -1$ has algebraic multiplicity 1.

<u>Basis for eigenspace $E_3(A)$ for $\lambda = 3$:</u>

We must find a basis for all solutions to $A\vec{x} = 3\vec{x}$.

Solving: $\begin{pmatrix} 3 & 0 \\ 8 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 3\begin{pmatrix} a \\ b \end{pmatrix}$

$$\begin{pmatrix} 3a + 0b \\ 8a - b \end{pmatrix} = \begin{pmatrix} 3a \\ 3b \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 8a - 4b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Need to solve:

$$\boxed{\begin{aligned} 8a - 4b &= 0 \\ 0 &= 0 \end{aligned}}$$

$\frac{1}{8}R_1 \to R_1$

or equivalently:

$$\boxed{\begin{aligned} a - \tfrac{1}{2}b &= 0 \\ 0 &= 0 \end{aligned}}$$

The solutions are:

$$b = t$$
$$a = \tfrac{1}{2}b = \tfrac{1}{2}t$$

So,

$$\vec{x} = \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \frac{1}{2}t \\ t \end{pmatrix} = t\begin{pmatrix} 1/2 \\ 1 \end{pmatrix}$$

gives all the elements in the eigenspace $E_3(A)$.

So, a basis for $E_3(A)$ is $\begin{pmatrix} 1/2 \\ 1 \end{pmatrix}$

and $\dim(E_3(A)) = 1$.

So, $\lambda = 3$ has geometric multiplicity 1.

Basis for eigenspace $E_{-1}(A)$ for $\lambda = -1$:

We need to solve $A\vec{x} = -\vec{x}$.

Solving:

$$\begin{pmatrix} 3 & 0 \\ 8 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = -\begin{pmatrix} a \\ b \end{pmatrix}$$

$$\begin{pmatrix} 3a + 0b \\ 8a - b \end{pmatrix} = \begin{pmatrix} -a \\ -b \end{pmatrix}$$

$$\begin{pmatrix} 4a \\ 8a \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Need to solve

$$\boxed{\begin{aligned} 4a &= 0 \\ 8a &= 0 \end{aligned}}$$

So, $a = 0$ and $b = t$ is free.

Thus,

$$\vec{x} = \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ t \end{pmatrix} = t\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

gives all the elements in the eigenspace $E_{-1}(A)$.

So, a basis for $E_{-1}(A)$ is $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\dim(E_{-1}(A)) = 1$ and the geometric multiplicity of $\lambda = -1$ is 1.

Summary table for $A = \begin{pmatrix} 3 & 0 \\ 8 & -1 \end{pmatrix}$

Eigenvalue λ	algebraic multiplicity	basis for $E_\lambda(A)$	geometric multiplicity
$\lambda = 3$	1	$\begin{pmatrix} 1/2 \\ 1 \end{pmatrix}$	1
$\lambda = -1$	1	$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$	1

① (b) $A = \begin{pmatrix} 10 & -9 \\ 4 & -2 \end{pmatrix}$

characteristic polynomial of A

$$\det(A - \lambda I_2) = \det\left(\begin{pmatrix} 10 & -9 \\ 4 & -2 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)$$

$$= \det \begin{pmatrix} 10-\lambda & -9 \\ 4 & -2-\lambda \end{pmatrix}$$

$$= (10-\lambda)(-2-\lambda) - (4)(-9)$$

$$= -20 - 10\lambda + 2\lambda + \lambda^2 + 36$$

$$= \lambda^2 - 8\lambda + 16$$

$$= (\lambda - 4)^2$$

Thus, $\lambda = 4$ is the only eigenvalue with algebraic multiplicity 2.

basis for $E_4(A)$ for eigenvalue $\lambda = 4$

We must solve $A\vec{x} = 4\vec{x}$.

Solving: $\begin{pmatrix} 10 & -9 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 4 \begin{pmatrix} a \\ b \end{pmatrix}$

$$\begin{pmatrix} 10a - 9b \\ 4a - 2b \end{pmatrix} = \begin{pmatrix} 4a \\ 4b \end{pmatrix}$$

$$\begin{pmatrix} 6a - 9b \\ 4a - 6b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

We must solve

$$\boxed{\begin{array}{l} 6a - 9b = 0 \\ 4a - 6b = 0 \end{array}}$$

We have

$$\left(\begin{array}{cc|c} 6 & -9 & 0 \\ 4 & -6 & 0 \end{array}\right) \xrightarrow{\frac{1}{6}R_1 \to R_1} \left(\begin{array}{cc|c} 1 & -3/2 & 0 \\ 4 & -6 & 0 \end{array}\right)$$

$$\xrightarrow{-4R_1 + R_2 \to R_2} \left(\begin{array}{cc|c} 1 & -3/2 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

We get:

$$\boxed{\begin{array}{r} a - \frac{3}{2}b = 0 \\ 0 = 0 \end{array}}$$

leading: a

free: b

So, $b = t$

$a = \frac{3}{2}b = \frac{3}{2}t$

Thus, all the elements of $E_4(A)$ are of the form

$$\vec{x} = \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 3/2\, t \\ t \end{pmatrix} = t \begin{pmatrix} 3/2 \\ 1 \end{pmatrix}$$

Thus, a basis for $E_4(A)$ is $\begin{pmatrix} 3/2 \\ 1 \end{pmatrix}$

So, $\dim(E_4(A)) = 1$ and the geometric multiplicity of $\lambda = 4$ is 1.

Summary table for A:

eigenvalue λ	algebraic multiplicity	basis for $E_\lambda(A)$	geometric multiplicity
$\lambda = 4$	2	$\begin{pmatrix} 3/2 \\ 1 \end{pmatrix}$	1

① (c) $A = \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}$

<u>characteristic polynomial of A</u>

$$\det(A - \lambda I_2) = \det\left(\begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix} - \lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)$$

$$= \det\begin{pmatrix} 5-\lambda & 0 \\ 0 & 5-\lambda \end{pmatrix}$$

$$= (5-\lambda)(5-\lambda) - (0)(0)$$

$$= (5-\lambda)(5-\lambda)$$

$$= (\lambda-5)(\lambda-5)$$

I factored out (−1) from each term and the two (−1)'s cancelled out

$$= (\lambda-5)^2$$

Thus, $\lambda = 5$ is the only eigenvalue of A and it has algebraic multiplicity 2.

<u>basis for $E_5(A)$ for $\lambda = 5$</u>:

We need to solve $A\vec{x} = 5\vec{x}$

Solving: $\begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 5\begin{pmatrix} a \\ b \end{pmatrix}$

$$\begin{pmatrix} 5a + 0b \\ 0a + 5b \end{pmatrix} = \begin{pmatrix} 5a \\ 5b \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

We get

$$\boxed{\begin{matrix} 0 = 0 \\ 0 = 0 \end{matrix}} \leftarrow$$

no leading variables
a, b are both free!

Solutions are:

$a = t$
$b = u$

Thus, all elements of $E_5(A)$ are of the form

$$\vec{x} = \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} t \\ u \end{pmatrix} = \begin{pmatrix} t \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ u \end{pmatrix} = t\begin{pmatrix} 1 \\ 0 \end{pmatrix} + u\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Thus, $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ span $E_5(A)$ and since they are linearly independent (it's the standard basis) they form a basis for $E_5(A)$.

Thus, $\dim(E_5(A)) = 2$ and the geometric multiplicity of $\lambda = 5$ is 2.

Summary table for $A = \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}$

eigenvalue λ	algebraic multiplicity	basis for $E_\lambda(A)$	geometric multiplicity
$\lambda = 5$	2	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$	2

(1)(d) $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$

characteristic poly of A

$$\det(A - \lambda I_3) = \det\left(\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right)$$

$$= \det\begin{pmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 2 \\ 0 & 0 & -\lambda \end{pmatrix} \qquad \begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

expand on col. 1

$$= (-\lambda)\underbrace{\begin{vmatrix} -\lambda & 2 \\ 0 & -\lambda \end{vmatrix}}_{\begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 2 \\ 0 & 0 & -\lambda \end{vmatrix}} - 0 + 0$$

$$= (-\lambda)\left[(-\lambda)(-\lambda) - (2)(0)\right]$$

$$= -\lambda^3 = -(\lambda - 0)^3$$

The only eigenvalue is $\lambda = 0$ and it has algebraic multiplicity 3.

<u>basis for $E_0(A)$ for $\lambda = 0$:</u>

Solving: $A\vec{x} = 0 \cdot \vec{x}$

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0\begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$\begin{pmatrix} b \\ 2c \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Need to solve:

$$\boxed{\begin{aligned} b \quad &= 0 \\ 2c &= 0 \\ 0 &= 0 \end{aligned}}$$

$$\left(\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{\frac{1}{2}R_2 \to R_2} \left(\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

giving:

$$\boxed{\begin{aligned} b \quad &= 0 \\ c &= 0 \\ 0 &= 0 \end{aligned}}$$

leading: b, c

free: c

Solution:

$a = t$

$b = 0$

$c = 0$

So every element $\vec{x}$ of $E_0(A)$ is of the form $\vec{x} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} t \\ 0 \\ 0 \end{pmatrix} = t\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

Thus, $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ is a basis for $E_0(A)$ and $\dim(E_0(A)) = 1$ and $\lambda = 0$ has geometric multiplicity 1.

Summary table for $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$

eigenvalue λ	algebraic multiplicity	basis for $E_\lambda(A)$	geometric multiplicity
$\lambda = 0$	3	$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$	1

① (e) $A = \begin{pmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ 1 & 0 & 4 \end{pmatrix}$

<u>characteristic poly for A</u>

$$\det(A - \lambda I_3) = \det\left(\begin{pmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ 1 & 0 & 4 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right)$$

$$= \det \begin{pmatrix} 4-\lambda & 0 & 1 \\ 2 & 3-\lambda & 2 \\ 1 & 0 & 4-\lambda \end{pmatrix}$$

expand on col 2

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

$$= -0 + (3-\lambda) \underbrace{\begin{vmatrix} 4-\lambda & 1 \\ 1 & 4-\lambda \end{vmatrix}}_{\begin{pmatrix} 4-\lambda & 0 & 1 \\ 2 & 3-\lambda & 2 \\ 1 & 0 & 4-\lambda \end{pmatrix}} - 0$$

$$= (3-\lambda)\left[(4-\lambda)(4-\lambda) - (1)(1)\right]$$

$$= (3-\lambda)\left[16 - 4\lambda - 4\lambda + \lambda^2 - 1\right]$$

$$= (3-\lambda)\left[\lambda^2 - 8\lambda + 15\right]$$

$$= (3-\lambda)(\lambda-3)(\lambda-5)$$

$$= -(\lambda-3)^2(\lambda-5)$$

So the eigenvalues are $\lambda = 3, 5$.
$\lambda = 3$ has algebraic multiplicity 2.
$\lambda = 5$ has algebraic multiplicity 1.

<u>basis for $E_3(A)$ for $\lambda = 3$:</u>

Solving: $A\vec{x} = 3\vec{x}$

$$\begin{pmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ 1 & 0 & 4 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 3\begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$\begin{pmatrix} 4a + c \\ 2a + 3b + 2c \\ a + 4c \end{pmatrix} = \begin{pmatrix} 3a \\ 3b \\ 3c \end{pmatrix}$$

$$\begin{pmatrix} a + c \\ 2a + 2c \\ a + c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

<u>Need to solve</u>

$$\boxed{\begin{aligned} a + c &= 0 \\ 2a + 2c &= 0 \\ a + c &= 0 \end{aligned}}$$

Solving:

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 2 & 0 & 2 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right) \xrightarrow[-R_1+R_3\to R_3]{-2R_1+R_2\to R_2} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

This gives

$$\boxed{\begin{array}{rl} a \quad + c &= 0 \\ 0 &= 0 \\ 0 &= 0 \end{array}}$$

leading: a
free: b, c

Solutions:

$b = t$
$c = u$
$a = -c = -u$

Thus, every $\vec{x}$ in $E_3(A)$ is of the form

$$\vec{x} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -u \\ t \\ u \end{pmatrix} = \begin{pmatrix} 0 \\ t \\ 0 \end{pmatrix} + \begin{pmatrix} -u \\ 0 \\ u \end{pmatrix}$$

$$= t\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + u\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

Thus, $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ span $E_3(A)$.

Since these two vectors are not multiples of each other they form a basis for $E_3(A)$. Thus, $\dim(E_3(A)) = 2$ and $\lambda = 3$ has geometric multiplicity 2.

basis for $E_5(A)$ for $\lambda = 5$:

Solving: $A\vec{x} = 5\vec{x}$

$$\begin{pmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ 1 & 0 & 4 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 5\begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$\begin{pmatrix} 4a + c \\ 2a + 3b + 2c \\ a + 4c \end{pmatrix} = \begin{pmatrix} 5a \\ 5b \\ 5c \end{pmatrix}$$

$$\begin{pmatrix} -a + c \\ 2a - 2b + 2c \\ a - c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Need to solve

$$\boxed{\begin{aligned} -a \phantom{{}-2b} + c &= 0 \\ 2a - 2b + 2c &= 0 \\ a \phantom{{}-2b} - c &= 0 \end{aligned}}$$

Solving: $\left(\begin{array}{ccc|c} -1 & 0 & 1 & 0 \\ 2 & -2 & 2 & 0 \\ 1 & 0 & -1 & 0 \end{array}\right) \xrightarrow{-R_1 \to R_1} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 2 & -2 & 2 & 0 \\ 1 & 0 & -1 & 0 \end{array}\right)$

$$\xrightarrow[-R_1+R_3\to R_3]{-2R_1+R_2\to R_2}\left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & -2 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{-\frac{1}{2}R_2\to R_2} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Need to solve:

$$\boxed{\begin{aligned} a \qquad -c &= 0 \\ b - 2c &= 0 \\ 0 &= 0 \end{aligned}}$$

leading: a, b

free: c

Solutions are:

$c = t$

$b = 2c = 2t$

$a = c = t$

Thus every $\vec{x}$ in $E_5(A)$ is of the form

$$\vec{x} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} t \\ 2t \\ t \end{pmatrix} = t \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

Thus, $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ is a basis for $E_5(A)$

and $\dim(E_5(A)) = 1$ and

$\lambda = 5$ has geometric multiplicity 1.

Summary table for $A = \begin{pmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ 1 & 0 & 4 \end{pmatrix}$

eigenvalue λ	algebraic multiplicity	basis for $E_\lambda(A)$	geometric multiplicity
$\lambda = 3$	2	$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$	2
$\lambda = 5$	1	$\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$	1

① (d) $A = \begin{pmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$

<u>characteristic polynomial of A</u>

$$\det(A - \lambda I_3) = \det\left(\begin{pmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right)$$

$$= \det \begin{pmatrix} 4-\lambda & 0 & 1 \\ -2 & 1-\lambda & 0 \\ -2 & 0 & 1-\lambda \end{pmatrix}$$

expand on col. 2

$$\boxed{\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}}$$

$$= -0 + (1-\lambda)\underbrace{\begin{vmatrix} 4-\lambda & 1 \\ -2 & 1-\lambda \end{vmatrix}}_{\begin{pmatrix} 4-\lambda & 0 & 1 \\ -2 & 1-\lambda & 0 \\ -2 & 0 & 1-\lambda \end{pmatrix}} - 0$$

$$= (1-\lambda)\left[(4-\lambda)(1-\lambda) - (1)(-2)\right]$$

$$= (1-\lambda)\left[4 - 4\lambda - \lambda + \lambda^2 + 2\right]$$

$$= (1-\lambda)\left[\lambda^2 - 5\lambda + 6\right]$$

$$= (1-\lambda)(\lambda-3)(\lambda-2)$$

$$= -(\lambda-1)(\lambda-2)(\lambda-3)$$

Thus, the eigenvalues are $\lambda = 1, 2, 3$
each with algebraic multiplicity 1.

basis for $E_1(A)$ for $\lambda = 1$:

Need to solve $A\vec{x} = 1 \cdot \vec{x}$

Solving: $\begin{pmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 1 \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

$$\begin{pmatrix} 4a + c \\ -2a + b \\ -2a + c \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$\begin{pmatrix} 3a + c \\ -2a \\ -2a \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Need to solve

$$\boxed{\begin{aligned} 3a + c &= 0 \\ -2a &= 0 \\ -2a &= 0 \end{aligned}}$$

$$\left(\begin{array}{ccc|c} 3 & 0 & 1 & 0 \\ -2 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 \end{array}\right) \xrightarrow{\frac{1}{3}R_1 \to R_1} \left(\begin{array}{ccc|c} 1 & 0 & 1/3 & 0 \\ -2 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 \end{array}\right) \xrightarrow[2R_1 + R_3 \to R_3]{2R_1 + R_2 \to R_2} \left(\begin{array}{ccc|c} 1 & 0 & 1/3 & 0 \\ 0 & 0 & 2/3 & 0 \\ 0 & 0 & 2/3 & 0 \end{array}\right)$$

$$\xrightarrow[3/2 R_3 \to R_3]{3/2 R_2 \to R_2} \left(\begin{array}{ccc|c} 1 & 0 & 1/3 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right) \xrightarrow{-R_2 + R_3 \to R_3} \left(\begin{array}{ccc|c} 1 & 0 & 1/3 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

This gives:

$$\boxed{\begin{aligned} a \quad\quad + \tfrac{1}{3}c &= 0 \\ c &= 0 \\ 0 &= 0 \end{aligned}}$$

leading: a, c

free: b

Solutions:

$b = t$

$c = 0$

$a = -\frac{1}{3}c = 0$

Thus, all the vectors $\vec{x}$ in $E_1(A)$ are of the form $\vec{x} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ t \\ 0 \end{pmatrix} = t \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.

So, $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ is a basis for $E_1(A)$ and $\dim(E_1(A)) = 1$ and $\lambda = 1$ has geometric multiplicity 1.

<u>basis for $E_2(A)$ for $\lambda = 2$:</u>

Solving: $A\vec{x} = 2\vec{x}$

$$\begin{pmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 2 \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$\begin{pmatrix} 4a + c \\ -2a+b \\ -2a + c \end{pmatrix} = \begin{pmatrix} 2a \\ 2b \\ 2c \end{pmatrix}$$

$$\begin{pmatrix} 2a + c \\ -2a-b \\ -2a - c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Need to solve:

$$\boxed{\begin{aligned} 2a \quad + c &= 0 \\ -2a - b \quad &= 0 \\ -2a \quad - c &= 0 \end{aligned}}$$

Solving:

$$\left(\begin{array}{ccc|c} 2 & 0 & 1 & 0 \\ -2 & -1 & 0 & 0 \\ -2 & 0 & -1 & 0 \end{array}\right) \xrightarrow[R_1+R_3 \to R_3]{R_1+R_2 \to R_2} \left(\begin{array}{ccc|c} 2 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow[-R_2 \to R_2]{\frac{1}{2}R_1 \to R_1} \left(\begin{array}{ccc|c} 1 & 0 & 1/2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

We get:

$$\boxed{\begin{aligned} a \quad + \tfrac{1}{2}c &= 0 \\ b - c &= 0 \\ 0 &= 0 \end{aligned}}$$

leading: a, b

free: c

Solution: $c = t$

$b = c = t$

$a = -\frac{1}{2}c = -\frac{1}{2}t$

Thus all the vectors $\vec{x}$ in $E_2(A)$ are of the form $\vec{x} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}t \\ t \\ t \end{pmatrix} = t\begin{pmatrix} -1/2 \\ 1 \\ 1 \end{pmatrix}$

So, $\begin{pmatrix} -1/2 \\ 1 \\ 1 \end{pmatrix}$ is a basis for $E_2(A)$ and $\dim(E_2(A)) = 1$ and $\lambda = 2$ has geometric multiplicity 1.

<u>basis for $E_3(A)$ for $\lambda = 3$:</u>

Solving: $A\vec{x} = 3\vec{x}$

$$\begin{pmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 3\begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$\begin{pmatrix} 4a & & +c \\ -2a+b & & \\ -2a & & +c \end{pmatrix} = \begin{pmatrix} 3a \\ 3b \\ 3c \end{pmatrix}$$

$$\begin{pmatrix} a & & +c \\ -2a-2b & & \\ -2a & & -2c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Need to solve

$$\boxed{\begin{array}{rrrl} a & & +c & = 0 \\ -2a & -2b & & = 0 \\ -2a & & -2c & = 0 \end{array}}$$

Solving:

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ -2 & -2 & 0 & 0 \\ -2 & 0 & -2 & 0 \end{array}\right) \xrightarrow[2R_1+R_3\to R_3]{2R_1+R_2\to R_2} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & -2 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{-\frac{1}{2}R_2\to R_2} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

This gives:

$$\boxed{\begin{aligned} a \quad\quad + c &= 0 \\ b - c &= 0 \\ 0 &= 0 \end{aligned}}$$

leading: a, b
free: c

Solution is

$c = t$

$b = c = t$

$a = -c = -t$

Thus, every $\vec{x}$ in $E_3(A)$ has the form

$$\vec{x} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -t \\ t \\ t \end{pmatrix} = t\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

So, $\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$ is a basis for $E_3(A)$ and $\dim(E_3(A)) = 1$ and geometric multiplicity of $\lambda = 3$ is 1.

Summary table for $A = \begin{pmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$

eigenvalue λ	algebraic multiplicity	basis for $E_\lambda(A)$	geometric multiplicity
$\lambda = 1$	1	$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$	1
$\lambda = 2$	1	$\begin{pmatrix} -1/2 \\ 1 \\ 1 \end{pmatrix}$	1
$\lambda = 3$	1	$\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$	1

(2) Suppose $\vec{x}$ is an eigenvalue of A with eigenvalue λ.

Then, $A\vec{x} = \lambda\vec{x}$.

So, $A^2\vec{x} = A(A\vec{x}) = A(\lambda\vec{x})$

$$= \lambda(A\vec{x})$$

$$= \lambda \cdot \lambda\vec{x}$$

$$= \lambda^2\vec{x}.$$

And, $A^3\vec{x} = A(A^2\vec{x}) = A(\lambda^2\vec{x})$

$$= \lambda^2(A\vec{x})$$

$$= \lambda^2(\lambda\vec{x})$$

$$= \lambda^3\vec{x}$$

Carry on in this fashion we will get that $A^n\vec{x} = \lambda^n\vec{x}$ for $n = 1, 2, 3, 4, \ldots$

③ Recall that

$$E_\lambda(A) = \{\vec{x} \mid A\vec{x} = \lambda\vec{x} \text{ and } \vec{x} \in \mathbb{R}^n\}$$

(i) Note that $A\vec{0} = \vec{0} = 0 \cdot \vec{0}$.
Thus, $\vec{0}$ is in $E_\lambda(A)$. ← Since $A\vec{0} = 0 \cdot \vec{0}$

(ii) Suppose $\vec{x}_1$ and $\vec{x}_2$ are in $E_\lambda(A)$.
Then, $A\vec{x}_1 = \lambda\vec{x}_1$ and $A\vec{x}_2 = \lambda\vec{x}_2$.
Thus,

matrix multiplication property

$$\begin{aligned} A(\vec{x}_1 + \vec{x}_2) &= A\vec{x}_1 + A\vec{x}_2 \\ &= \lambda\vec{x}_1 + \lambda\vec{x}_2 \\ &= \lambda(\vec{x}_1 + \vec{x}_2) \end{aligned}$$

vector scaling property

Thus, $\vec{x}_1, \vec{x}_2$ are in $E_\lambda(A)$. ← since $A(\vec{x}_1 + \vec{x}_2) = \lambda(\vec{x}_1 + \vec{x}_2)$

(iii) Suppose $\vec{x}_3$ is in $E_\lambda(A)$ and α is a real number.

Then, $\boxed{A\vec{x}_3 = \lambda \vec{x}_3}$ since $\vec{x}_3 \in E_\lambda(A)$.

So,

matrix multiplication property

$$A(\alpha \vec{x}_3) = \alpha(A\vec{x}_3)$$
$$= \alpha(\lambda \vec{x}_3)$$
$$= \lambda(\alpha \vec{x}_3)$$

Thus, $A(\alpha \vec{x}_3) = \lambda(\alpha \vec{x}_3)$

So, $\alpha \vec{x}_3$ is in $E_\lambda(A)$.

By (i), (ii), (iii), we know that $E_\lambda(A)$ is a subspace of $\mathbb{R}^n$.

$\blacksquare$